On Grassmannian description of the constrained KP hierarchy

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Abstract

This note develops an explicit construction of the constrained KP hierarchy within the Sato Grassmannian framework. Useful relations are established between the kernel elements of the underlying ordinary differential operator and the eigenfunctions of the associated KP hierarchy as well as between the related bilinear concomitant and the squared eigenfunction potential. © 1999 Elsevier Science B.V. All rights reserved.

1. Introduction

The purpose of this note is to present construction of the constrained KP (cKP) hierarchy within the Sato Grassmannian context using elements of the kernels of the underlying differential operators. The fundamental concept is the canonical pairing (the bilinear concomitant) introduced here on the space of elements of the kernels of the underlying differential operator and its conjugated counterpart. The formalism is simplified by relations between the bilinear concomitant and the squared eigenfunction potential (SEP) which emerged before in the setting of the KP hierarchy [1–3]. The claim is that use of SEP makes construction of the cKP hierarchy within the Sato Grassmannian theory of the KP hierarchy more transparent. The cKP hierarchy has recently been discussed in [4–7] using Segal–Wilson modification of the Sato Grassmannian. This note provides a link between these works and the current formalism based on the Sato Grassmannian and the SEP method. Our approach emphasizes connection between elements of kernels of differential operators and eigenfunctions (see

below for the definition) of the pseudo-differential operators realized as ratios of the under-lying differential operators. These kernel elements yield a convenient parameterization for the "coordinate-free" characterization of cKP hierarchy developed in [6,7].

2. KP hierarchy

We first briefly review the KP hierarchy of nonlinear evolution equations in the approach based on the calculus of the pseudo-differential operators. The main object here is the pseudo-differential Lax operator L:

$$L = D^r + \sum_{j=0}^{r-2} v_j D^j + \sum_{i \geq 1} u_i D^{-i}. \tag{2.1}$$

The operator D satisfies the generalized Leibniz rule, so for instance $[D, f] = f'$ with $f' = \partial_x f = \partial f / \partial x$.

The associated isospectral flows are described by the Lax equations:

$$\frac{\partial}{\partial t_n} L = [L_+^{n/r}, L], \quad n = 1, 2, \ldots, \tag{2.2}$$

with $x \equiv t_1$. In (2.2) and below, the subscripts ($\pm$) of pseudo-differential operators in-dicate projections on purely differential/pseudo-differential parts. The Zakharov–Shabat zero-curvature equations ensure commutativity of the isospectral flows $\partial / \partial t_n$ in (2.2).

For a given Lax operator L, which satisfies Sato's flow equation (2.2), the function Φ (Ψ), whose flows obey:

$$\frac{\partial \Phi}{\partial t_l} = L_+^{l/r}(\Phi), \qquad \frac{\partial \Psi}{\partial t_l} = -(L^*)_+^{l/r}(\Psi), \quad l = 1, 2, \ldots, \tag{2.3}$$

is called an *(adjoint) eigenfunction* of L. In (2.3) we have introduced an operation of conjugation, defined by simple rules $D^* = -D$ and $(AB)^* = B^*A^*$. Throughout this paper we will follow the convention that for any (pseudo-)differential operator A and a function f, the symbol $A(f)$ will indicate application (action) of A on f while the symbol Af will be just a product of A with the zero-order (multiplication) operator f.

One can also represent the Lax operator in terms of the dressing operator $W = 1 + \sum_1^\infty w_n D^{-n}$ through $L = W D^r W^{-1}$. In this framework, Eq. (2.2) is equivalent to the so-called Wilson–Sato equation:

$$\partial_n W = -(W D^n W^{-1})_- W, \tag{2.4}$$

where $\partial_n = \partial / \partial t_n$. Next, we define corresponding wave-eigenfunction via

$$\psi_W(t, \lambda) = W(e^{\xi(t,\lambda)}) = \left(1 + \sum_{i=1}^\infty w_i(t) \lambda^{-i}\right) e^{\xi(t,\lambda)}, \tag{2.5}$$

where

$$\xi(t,\lambda) \equiv \sum_{n=1}^{\infty} t_n \lambda^n, \quad t_1 = x. \tag{2.6}$$

Similarly, there is also an adjoint wave-eigenfunction:

$$\psi_W^* = W^{*-1}(e^{-\xi(t,\lambda)}) = \left(1 + \sum_{i=1}^{\infty} w_i^*(t)\lambda^{-i}\right) e^{-\xi(t,\lambda)}. \tag{2.7}$$

As seen from (2.4) and (2.5), the wave-eigenfunction is an eigenfunction which in addition to the first of Eqs. (2.3) also satisfies the spectral equations $L\psi_W(\lambda,t) = \lambda^r \psi_W(\lambda,t)$. The wave-eigenfunction and its adjoint enter the fundamental Hirota's bilinear identity:

$$\int d\lambda \, \psi_W^*(t,\lambda)\psi_W(t',\lambda) = 0, \tag{2.8}$$

which generates the entire KP hierarchy via Hirota's equations for the underlying tau-functions (see e.g. [8]). In (2.8) and in what follows integrals over spectral parameters denote: $\int d\lambda \equiv \oint_0 d\lambda/2i\pi = \text{Res}_{\lambda=0}$. The proper understanding of (2.8) requires, following e.g. [8], expanding of $\psi_W(t',\lambda)$ in (2.8) as formal power series w.r.t. $t'_n - t_n, n = 1, 2, \ldots$, according to

$$\psi_W(t',\lambda) = \sum \frac{(t'_1 - t_1)^{k_1} \cdots (t'_n - t_n)^{k_n}}{k_1! \cdots k_n!} \partial_1^{k_1} \cdots \partial_n^{k_n} \psi_W(t,\lambda). \tag{2.9}$$

The wave function is an oscillatory function of order 0. Generally, the oscillatory function of order l is of the form:

$$f(t,\lambda)e^{\xi(t,\lambda)} = \left(\lambda^l + \sum_{j<l} a_j(t)\lambda^j\right) e^{\xi(t,\lambda)}. \tag{2.10}$$

It will be of importance for us that the action of the differential operator D can be inverted uniquely on the space of oscillatory functions according to

$$D^{-1}f(t,\lambda)\,e^{\xi(t,\lambda)} = \sum_{\alpha=0}^{\infty}(-1)^\alpha f^{(\alpha)}(t,\lambda)\lambda^{-1-\alpha}\,e^{\xi(t,\lambda)}. \tag{2.11}$$

Consider now the one-form: $\omega = \sum_n \text{Res}(D^{-1}\Psi L_+^{n/r}\Phi D^{-1})dt_n$ defined for a couple of (adjoint) eigenfunctions Φ and Ψ. As in [1,2] one can show using the Zakharov–Shabat equations that ω is a closed form with respect to exterior derivative $d \equiv \sum_n \partial_n dt_n$. By the usual argument one concludes from $d\omega = 0$ that the one form ω can be rewritten as $\omega = dS(\Phi, \Psi)$. This procedure defines (up to a constant) a squared eigenfunction potential (SEP) $S(\Phi, \Psi)$. In components, the flows of $S(\Phi, \Psi)$ are given by

$$\partial_n S(\Phi, \Psi) = \text{Res}(D^{-1}\Psi L_+^{n/r}\Phi D^{-1}). \tag{2.12}$$

Especially, $\partial_x S(\Phi(t), \Psi(t)) = \Phi(t)\Psi(t)$. As shown in [3] the squared eigenfunction potential defines a spectral representation of (adjoint) eigenfunctions. The statement is

as follows. Any (adjoint) eigenfunction of the general KP hierarchy possesses a spectral representation:

$$\Phi(t) = -\int d\lambda \, \psi_W(t, \lambda) \, S(\Phi(t'), \psi_W^*(t', \lambda)), \tag{2.13}$$

$$\Psi(t) = \int d\lambda \, \psi_W^*(t, \lambda) \, S(\psi_W(t', \lambda), \Psi(t')), \tag{2.14}$$

with spectral densities given by the squared eigenfunction potentials at some multi-time $t' = (t_1', t_2', \ldots)$ taken at some arbitrary fixed value. The r.h.s. of (2.13) and (2.14) do not depend on t'. Furthermore, the closed expressions have been found in [3] for those squared eigenfunction potentials which have as argument at least one oscillating wave-eigenfunction:

$$S(\Phi(t), \psi_W^*(t, \lambda)) = -\frac{1}{\lambda} \psi_W^*(t, \lambda) \Phi(t + [\lambda^{-1}]), \tag{2.15}$$

$$S(\psi_W(t, \lambda), \Psi(t)) = \frac{1}{\lambda} \psi_W(t, \lambda) \Psi(t - [\lambda^{-1}]). \tag{2.16}$$

In the above equation $S(\psi_W(t, \lambda), \Psi(t))$ is the squared eigenfunction potential (SEP) associated with a pair of eigenfunctions $\psi_W(t, \lambda)$ and $\Psi(t)$. It is an oscillatory function of order -1:

$$S(\psi_W(t, \lambda), \Psi(t)) = \sum_{j=1}^{\infty} s_j(t)\lambda^{-j} \, e^{\xi(t, \lambda)} = [\Psi(t)\lambda^{-1} + O(\lambda^{-2})] e^{\xi(t, \lambda)}. \tag{2.17}$$

We will now make connection to the language of the universal Sato Grassmannian. Consider the hyperplane **W** defined through a linear basis of Laurent series $\{f_k(\lambda)\}$ in λ in terms of the wave-eigenfunction as a generating function:

$$\mathbf{W} \equiv \mathrm{span}\langle f_0(\lambda), \, f_1(\lambda), \, f_2(\lambda), \ldots \rangle$$

$$f_k(\lambda) = \frac{\partial^k}{\partial x^k} \psi_W(t, \lambda) \Big|_{x=t_2=t_3=\cdots=0} \tag{2.18}$$

Obviously, **W** is closed under the differentiation $\partial/\partial x$. From the fact that $\psi_W(t, \lambda)$ satisfies Eq. (2.3) we obtain an alternative definition of **W**:

$$\mathbf{W} = \mathrm{span}\{\psi_W(t, \lambda), \text{ all } t \in \mathbb{C}^\infty\} \tag{2.19}$$

A typical element of **W**; $f_k(\lambda) = (\lambda^k + O(\lambda^{k-1})) \exp \xi(t, \lambda)$, has an order $k \geq 0$. Consequently, the set of orders of all elements of **W** is given by the set of non-negative integers.

In case of the standard rth KdV reduction, the corresponding Lax operator $\mathcal{L} = D + \sum_1^\infty u_i D^{-i}$ satisfies the constraint $\mathcal{L}^r = \mathcal{L}_+^r$, which translates to the Grassmannian language as $\lambda^r \mathbf{W} \subset \mathbf{W}$.

It is clear that $\int d\lambda \, \psi_W^*(t, \lambda)\chi(t', \lambda) = 0$ for any $\chi(t, \lambda) \in \mathbf{W}$. We will make here a plausible assumption that the inverse holds as well. More precisely, the statement is as follows. Let $F(\psi_W(t', \lambda))$ be a linear functional of $\psi_W(t, \lambda)$ of a positive order for which the following bilinear equation $\int d\lambda \, \psi_W^*(t, \lambda)F(\psi_W(t', \lambda)) = 0$ holds for all t', then $F(\psi_W(t', \lambda)) \in \mathbf{W}$.

3. Differential operators and the canonical pairing structure

Consider a differential operator of order m:

$$L_m = D^m + u_{m-1} D^{m-1} + \cdots + u_1 D^1 + u_0. \tag{3.1}$$

The differential operator of order m is called *monic* if its leading term is D^m. A monic differential operator of order m is fully characterized by m elements of its kernel. For instance, let functions ϕ_i, $i = 1, \ldots, m$ constitute a basis for $\operatorname{Ker} L_m = \{\phi_1, \ldots, \phi_m\}$, then

$$L_m(f) = \frac{W_{m+1}[\phi_1, \ldots, \phi_m, f]}{W_m[\phi_1, \ldots, \phi_m]}. \tag{3.2}$$

The elements of $\operatorname{Ker} L_m$ are assumed to be linearly independent so that the Wronskian matrix

$$(\mathcal{W}_{m \times m})_{1 \leq i, j \leq m} = \partial_x^{j-1} \phi_i \tag{3.3}$$

is invertible. In different words the Wronskian determinant $W_m[\phi_1, \ldots, \phi_m] = \det \|\mathcal{W}_{m \times m}\|$ must be different from zero. Let $(\mathcal{W}_{m \times m}^{-1})_{ij}$, $i, j = 0, \ldots, m-1$, be the matrix elements of the inverse of the Wronskian matrix $\mathcal{W}_{m \times m}$. The following relations are then satisfied:

$$\sum_{j=1}^{m} (\mathcal{W}_{m \times m}^{-1})_{ij} \phi_k^{(j-1)} = \delta_{i,k}, \qquad \sum_{k=1}^{m} \phi_k^{(j-1)} (\mathcal{W}_{m \times m}^{-1})_{kl} = \delta_{j,l}, \tag{3.4}$$

and

$$(\mathcal{W}_{m \times m}^{-1})_{ij} = (-1)^{i+j} \frac{\det_{(j,i)} \|\mathcal{W}_{m \times m}\|}{W_m[\phi_1, \ldots, \phi_m]}, \tag{3.5}$$

where the determinant on the right-hand side is the minor determinant obtained by extracting the jth row and ith column from the Wronski matrix $\mathcal{W}_{m \times m}$ given in Eq. (3.3).

The following technical identity:

$$\sum_{j=1}^{m} (\mathcal{W}_{m \times m}^{-1})_{ij} \chi^{(j-1)}$$

$$= (-1)^{m+i} \frac{W_m[\phi_1, \ldots, \widehat{\phi}_i, \ldots, \phi_m, \chi]}{W_m[\phi_1, \ldots, \phi_m]}, \quad i = 1, \ldots, m, \tag{3.6}$$

which is valid for an arbitrary function χ, follows directly from (3.3)–(3.5).

In addition, we also need to consider an adjoint operator L_m^* obtained from (3.1) by a process of conjugation described below Eq. (2.3). Let $\psi_1, \ldots, \psi_m$ be elements of the kernel of an adjoint operator L_m^*:

$$\operatorname{Ker} L_m^* = \{\psi_1, \ldots, \psi_m\}. \tag{3.7}$$

They are given in terms of elements of $\operatorname{Ker} L_m$ via relation [9,10]:

$$\psi_i = (-1)^{m+i} \frac{W_{m-1}[\phi_1, \ldots, \widehat{\phi}_i, \ldots, \phi_m]}{W_m[\phi_1, \ldots, \phi_m]}, \quad i = 1, \ldots, m. \tag{3.8}$$

Comparing with (3.5) we see that relation (3.8) expresses the fact that $(\psi_1, \ldots, \psi_m)^{\mathrm{T}}$ is the last column in the inverse $\mathcal{W}^{-1}$ of the Wronskian matrix $\mathcal{W}$ of $(\phi_1, \ldots, \phi_m)$. In particular, we see that the functions $\{\psi_1, \ldots, \psi_m\}$ are also linearly independent.

Some of the obvious consequences of definition (3.8) and connection between functions ψ_i and the matrix $\mathcal{W}^{-1}$ are:

$$\sum_{i=1}^{m} \phi_i^{(k)}(t)\psi_i(t) = \delta_{k,m-1} \quad \text{for } k = 0, 1, \ldots, m-1. \tag{3.9}$$

For completeness let us list the extension of (3.9) to $k = m$:

$$\sum_{i=1}^{m} \phi_i^{(m)}(t)\psi_i(t) = \sum_{i=1}^{m}(-1)^{m-i} \frac{W_{m-1}[\phi_1, \ldots, \widehat{\phi_i}, \ldots, \phi_m]\phi_i^{(m)}}{W_m[\phi_1, \ldots, \phi_m]}$$

$$= \partial_x \ln W_m[\phi_1, \ldots, \phi_m]. \tag{3.10}$$

Consider now the quantity $N \equiv \sum_{i=1}^{m} \phi_i\, D^{-1}\, \psi_i$. It follows easily that

$$(L_m N)_- = \sum_{i=1}^{m} L_m(\phi_i)\, D^{-1}\, \psi_i = 0. \tag{3.11}$$

Moreover, using the Leibniz rule we obtain from (3.9) and (3.10):

$$(L_m N)_+ = \left(L_m \left(\sum_{i=1}^{m} \sum_{\alpha=1}^{\infty} D^{-1-\alpha} \phi_i^{(\alpha)}\, \psi_i \right) \right)_+$$

$$= (L_m(D^{-m} + D^{-1-m}\partial_x \ln W_m[\phi_1, \ldots, \phi_m] + O(D^{-2-m})))_+ = 1. \tag{3.12}$$

Hence, as in [9] we obtain from (3.11) and (3.12):

$$L_m^{-1} = \sum_{i=1}^{m} \phi_i\, D^{-1}\, \psi_i. \tag{3.13}$$

Consider now $\mathrm{Res}(D^{-1}\psi_j L_m \sum_{i=1}^{m} \phi_i\, D^{-1}\, \psi_i)$. In view of (3.13), we find

$$\mathrm{Res}\left(D^{-1}\psi_j L_m \sum_{i=1}^{m} \phi_i\, D^{-1}\, \psi_i \right) = \psi_j = \sum_{i=1}^{m} \mathrm{Res}(D^{-1}\psi_j L_m \phi_i\, D^{-1})\psi_i. \tag{3.14}$$

One notices that

$$\partial_x \mathrm{Res}(D^{-1}\psi_j L_m \phi_i\, D^{-1}) = \psi_j L_m(\phi_i) + L_m^*(\psi_j)\, \phi_i = 0, \tag{3.15}$$

and therefore $\mathrm{Res}(D^{-1}\psi_j L_m \phi_i\, D^{-1})$ is a constant in x. Since functions ψ_i are linearly independent, we conclude in view of Eqs. (3.14) and (3.15) that:

$$\mathrm{Res}(D^{-1}\psi_j\, L_m \phi_i\, D^{-1}) = \delta_{i,j}. \tag{3.16}$$

It appears, therefore, that $\{\psi_1, \ldots, \psi_m\}$ can be viewed as dual basis of $\{\phi_1, \ldots, \phi_m\}$ with respect to a canonical pairing defined in terms of the so-called bilinear concomitant (see [11–13]):

$$\langle \phi | \psi \rangle_{L_m} \equiv \mathrm{Res}(D^{-1}\psi L_m \phi D^{-1}) \tag{3.17}$$

$$= \sum_{i=1}^{m} \sum_{j=0}^{i-1} (-1)^j \phi^{(i-j-1)} (u_i \psi)^{(j)} \tag{3.18}$$

with $u_m = 1$. In this setting the bases $\{\phi_1, \ldots, \phi_m\}$ and $\{\psi_1, \ldots, \psi_m\}$ related through (3.8) are dual to each other in the sense of satisfying $\langle \phi_i | \psi_j \rangle_{L_m} = \delta_{ij}$ for $i, j = 1, \ldots, m$ due to (3.16). The following technical lemma provides a useful characterization of the products $\langle \chi | \psi_i \rangle_{L_m}$ for an arbitrary function χ and $\psi_i \in \mathrm{Ker}\, L_m^*$.

Lemma 3.1. *The following identity:*

$$\langle \chi | \psi_i \rangle_{L_m} = \sum_{j=1}^{m} (W_{m \times m}^{-1})_{ij} \chi^{(j-1)}, \quad i = 1, \ldots, m, \tag{3.19}$$

holds for an arbitrary function χ and $\psi_i \in \mathrm{Ker}\, L_m^*$.

Proof. Let $M_i = \sum_{j=1}^{m} (W_{m \times m}^{-1})_{ij} D^j$ be $(m-1)$-order differential operator for the fixed i. We know its $m-1$ null-functions ϕ_k such that $M_i(\phi_k) = 0$ for $k \neq i$. We also have $M_i(\phi_i) = 1$. This characterizes M_i completely. Note, that the $(m-1)$-order differential operator $\langle \cdot | \psi_i \rangle_{L_m}$ agrees with M_i on ϕ_i, $i = 1, \ldots, m$, which completes the proof. □

Recalling identity (3.6) we find an alternative way of writing (3.19) as

$$\langle \chi | \psi_i \rangle_{L_m} = (-1)^{m+i} \frac{W_m[\phi_1, \ldots, \widehat{\phi_i}, \ldots, \phi_m, \chi]}{W_m[\phi_1, \ldots, \phi_m]}, \tag{3.20}$$

from which the next corollary follows.

Corollary 3.1.

$$\langle \chi | \psi_i \rangle_{L_m} = \psi_i L_{m,i}(\chi), \tag{3.21}$$

where $L_{m,i}$ are the ordinary differential operators of order $m-1$, whose kernels are given by $\mathrm{Ker}(L_{m,i}) = \{\phi_1, \ldots, \widehat{\phi_i}, \ldots, \phi_m\}$. Correspondingly, the action of $L_{m,i}$ is defined through

$$L_{m,i}(\chi) \equiv \frac{W_m[\phi_1, \ldots, \widehat{\phi_i}, \ldots, \phi_m, \chi]}{W_{m-1}[\phi_1, \ldots, \widehat{\phi_i}, \ldots, \phi_m]}. \tag{3.22}$$

We will now introduce isospectral deformations of the differential operator L_m of the form:

$$\partial_n L_m = \widetilde{B}_n L_m - L_m B_n, \quad n = 1, 2, \ldots \tag{3.23}$$

In this setting we will show that the product $\langle \cdot | \cdot \rangle_{L_m}$ defines a canonical pairing Ker $L_m \times$ Ker $L_m^* \to \mathbb{C}$. As discussed in [12,13] this pairing is nonsingular.

The two families of differential operators $\widetilde{B}_n$ and B_n are both assumed to satisfy Zakharov–Shabat equations:

$$0 = \partial_k \widetilde{B}_n - \partial_n \widetilde{B}_k + [\, \widetilde{B}_n \, , \, \widetilde{B}_k \,],$$
$$0 = \partial_k B_n - \partial_n B_k + [\, B_n \, , \, B_k \,], \quad k, n = 1, 2, \ldots, \tag{3.24}$$

to ensure commutativity of flows defined in (3.23). From (3.23), we find that

$$\partial_n L_m^{-1} = B_n L_m^{-1} - L_m^{-1} \widetilde{B}_n, \quad n = 1, 2, \ldots \tag{3.25}$$

The following result applies to this case [10] .

Lemma 3.2. *Eqs. (3.23) imply that $\phi_i \in$ Ker L_m and $\psi_i \in$ Ker L_m^* are "up to a gauge rotation" (adjoint) eigenfunctions satisfying:*

$$\partial_n \phi_i = B_n(\phi_i), \quad i = 1, \ldots, m, \tag{3.26}$$
$$\partial_n \psi_i = -\widetilde{B}_n^*(\psi_i), \quad i = 1, \ldots, m. \tag{3.27}$$

Proof. From $\partial_n L_m(\phi_i) = 0$ and (3.23) we find that $B_n(\phi_i) - \partial_n \phi_i \in$ Ker L_m. Hence we can write

$$B_n(\phi_i) - \partial_n \phi_i = -\sum_{j=1}^{m} \phi_j \, c_{ji}^{(n)}(\tilde{t}), \tag{3.28}$$

where $\tilde{t} = (t_2, t_3, \ldots)$. We now proceed in a way similar to the one used, in a slightly different setting, in [14]. Define $(\Delta_n)_{jk} \equiv \partial_n \delta_{jk} - c_{kj}^{(n)}$ so that we can compactly rewrite (3.28) as $(\Delta_n)_{jk} \phi_k = B_n(\phi_j)$. The Zakharov–Shabat equations (3.24), ensure the zero curvature equation $([\, \Delta_n \, , \, \Delta_l \,])_{ik} \phi_k = 0$. Thus the "connection" $c_{ij}^{(n)}$ is a pure gauge and can be cast in a form

$$c_{ij}^{(n)}(\tilde{t}) = (c^{-1})_{ik}(\tilde{t}) \, \partial_n c_{kj}(\tilde{t}), \quad n \geq 2. \tag{3.29}$$

Define accordingly

$$\bar{\phi}_j \equiv \phi_k \, (c^{-1})_{kj} \tag{3.30}$$

It is easy to verify that $\bar{\phi}_j$ satisfy

$$\partial_n \bar{\phi}_j = (\Delta_n \phi)_k (c^{-1})_{kj} = B_n(\bar{\phi}_j). \tag{3.31}$$

Similarly, from $\partial_n L_m^*(\psi_i) = 0$, we arrive at

$$\widetilde{B}_n^*(\psi_i) + \partial_n \psi_i = \sum_{j=1}^{m} \bar{c}_{ij}^{(n)}(\tilde{t}) \, \psi_j. \tag{3.32}$$

We will now establish a relation between coefficients $c_{ij}^{(n)}$ $\bar{c}_{ij}^{(n)}$. We need at this point the technical identity:

$$[K, fD^{-1}g]_- = K(f)D^{-1}g - fD^{-1}K^*(g) \tag{3.33}$$

valid for a purely differential operator K and arbitrary functions f, g. We find from (3.25) and the above equation that

$$(\partial_n L_m^{-1})_- = \sum_{i=1}^m (B_n(\phi_i))D^{-1}\psi_i - \sum_{i=1}^m \phi_i D^{-1}\widetilde{B}_n^*(\psi_i). \tag{3.34}$$

Eqs. (3.28) and (3.32) agree with (3.34) provided

$$\sum_{i,j=1}^m (c_{ij}^{(n)} + \bar{c}_{ij}^{(n)})\phi_i D^{-1}\psi_j = 0. \tag{3.35}$$

Define a differential operator of $(m-1)$ order:

$$K[\phi] \equiv \sum_{s=1}^m \sum_{l=0}^{s-1} u_l D^l (\phi)^{(s-l-1)} \tag{3.36}$$

such that $K^*[\phi](\psi) = \langle \phi | \psi \rangle_{L_m}$. From (3.35) and (3.16), we find

$$\left(\sum_{i,j=1}^m (c_{ij}^{(n)} + \bar{c}_{ij}^{(n)})\phi_i D^{-1}\psi_j K[\phi_k] \right)_- = 0 \tag{3.37}$$

or

$$\sum_{i=1}^m (c_{ik}^{(n)} + \bar{c}_{ik}^{(n)})\phi_i = 0, \quad k = 1, \ldots, m. \tag{3.38}$$

Since $\{\phi_i\}$ are linearly independent we find from (3.38) that $c_{ij}^{(n)} = -\bar{c}_{ij}^{(n)}$ for all $i, j = 1, \ldots, m$. Accordingly, $(\Delta_n^*)_{jk}\psi_k = -(\mathcal{L}_{m+1}^*)_+^{n/r}(\psi_j)$ with $(\Delta_n^*)_{jk} \equiv \partial_n \delta_{jk} + c_{jk}^{(n)}$. Define next

$$\bar{\psi}_j \equiv c_{jk}\psi_k. \tag{3.39}$$

It follows that

$$\partial_n \bar{\psi}_j = c_{jk}(\Delta_n^*\psi)_k = -\widetilde{B}_n^*(\bar{\psi}_j). \tag{3.40}$$

Hence we succeeded to find a mutually inverse gauge rotation taking $\phi_i \in \text{Ker } L_m$ and $\psi_i \in \text{Ker } L_m^*$ into (adjoint) eigenfunctions satisfying (3.26) and (3.27). □

Lemma 3.3. *Let ϕ and ψ satisfy Eqs. (3.26) and (3.27) with respect to flows entering Eq. (3.23), then*

$$\partial_n \langle \phi | \psi \rangle_{L_m} = \text{Res}(D^{-1}\psi \widetilde{B}_n L_m(\phi)D^{-1}) - \text{Res}(D^{-1}L_m^*(\psi)B_n\phi D^{-1})$$
$$= \langle L_m(\phi)|\psi \rangle_{\widetilde{B}_n} - \langle \phi | L_m^*(\psi) \rangle_{B_n}, \quad n = 1, 2, \ldots \tag{3.41}$$

Proof. Proof follows from the technical lemma [1]:

$$\mathrm{Res}(D^{-1}L_1L_2D^{-1}) = \mathrm{Res}(D^{-1}(L_1^*)_0L_2D^{-1}) + \mathrm{Res}(D^{-1}L_1(L_2)_0D^{-1}), \quad (3.42)$$

where L_1 and L_2 are arbitrary differential operators and $(\cdot)_0$ denotes projection on the zero-order term. With help of relation (3.42) we can rewrite $\mathrm{Res}(D^{-1}\psi L_m B_n\phi D^{-1})$ as a sum of $\mathrm{Res}(D^{-1}L_m^*(\psi)B_n\phi D^{-1})$ and $\mathrm{Res}(D^{-1}\psi L_m B_n(\phi)D^{-1})$. $\quad\square$

Corollary 3.2. *For $\phi \in \mathrm{Ker}\, L_m$ and $\psi \in \mathrm{Ker}\, L_m^*$ and L_m satisfying Eq. (3.23) it holds that $\partial_n \langle\phi|\psi\rangle_{L_m} = 0$ for $n = 1, 2, \ldots$ Accordingly, $\langle\cdot|\cdot\rangle_{L_m}$ defines a canonical pairing $\mathrm{Ker}\, L_m \times \mathrm{Ker}\, L_m^* \to \mathbb{C}$.*

As a special case $(n = 1)$ of Lemma 3.3 we obtain a basic relation:

$$\partial_x \langle\phi|\psi\rangle_{L_m} = L_m(\phi)\,\psi - \phi\, L_m^*(\psi). \qquad (3.43)$$

Note that the result (3.43) is valid this time for an arbitrary ϕ, ψ as follows by a direct verification.

Another consequence of Lemma 3.3 reads as follows.

Corollary 3.3. *For ϕ, ψ satisfying condition of Lemma 3.3 and the differential operator L_m whose isospectral flows are given in (3.23) the following relation:*

$$\langle\phi|\psi\rangle_{L_m} = S(L_m(\phi), \psi) - S(\phi, L_m^*(\psi)) \qquad (3.44)$$

holds up to a constant (in the multi-time t).

Eq. (3.44) follows easily from Eq. (3.41) and relations:

$$\partial_n S(L_m(\phi), \psi) = \langle L_m(\phi)|\psi\rangle_{\widetilde{B}_n}, \qquad \partial_n S(\phi, L_m^*(\psi)) = \langle\phi|L_m^*(\psi)\rangle_{B_n}. \qquad (3.45)$$

Corollary 3.3 establishes a relation between the bilinear concomitant and the square eigenfunction potential.

4. Lax operator representation of the cKP hierarchy

We are studying here the class of constrained cKP hierarchies for which we have the Lax representation:

$$L = D^r + \sum_{l=0}^{r-2} u_l D^l + \sum_{i=1}^{m} \Phi_i D^{-1}\Psi_i = B_r + \sum_{i=1}^{m} \Phi_i D^{-1}\Psi_i \qquad (4.1)$$

with Φ_i and Ψ_i being (adjoint) eigenfunctions of the Lax operator L as in (2.3). As shown in [6,7,10] the cKP$_{r,m}$ hierarchy can be expressed in terms of two normalized differential

operators L_m and L_{m+r} of order m and $r + m$, respectively. The Lax operator (4.1) of the cKP $_{r,m}$ hierarchy is in this representation being rewritten as a ratio:

$$L \equiv L_m^{-1} L_{m+r} = \sum_{i=1}^{m} \phi_i D^{-1} L_{m+r}^*(\psi_i) + B_r. \tag{4.2}$$

The wave eigenfunction $\psi_W(t, \lambda)$ of (4.1) is an eigenfunction (as in Eq. (2.3)) which additionally satisfies the following spectral equation:

$$L \psi_W(t, \lambda) \equiv B_r(\psi_W(t, \lambda)) + \sum_{i=1}^{m} \Phi_i(t) S(\psi_W(t, \lambda), \Psi_i(t)) = \lambda^r \psi_W(t, \lambda). \tag{4.3}$$

In Eq. (4.3), $S(\psi_W(t, \lambda), \Psi_i(t))$ is the squared eigenfunction potential (SEP) associated with a pair of eigenfunctions $\psi_W(t, \lambda)$ and $\Psi_i(t)$.

5. Universal Sato's Grassmannian construction of the cKP Hierarchy

Let us first introduce the following basic definition.

Definition 5.1. For the wave-eigenfunction $\psi_W(t, \lambda)$ of the KP hierarchy and $\psi_i \in \mathrm{Ker} L_m^*$, we define m objects:

$$S_i(t, \lambda) \equiv \lambda^r \langle \psi_W | \psi_i \rangle_{L_m}, \quad i = 1, \ldots, m. \tag{5.1}$$

As seen from Eq. (3.43), these m objects $S_i(t, \lambda)$ defined in (5.1) satisfy

$$\partial_x S_i(t, \lambda) = \lambda^r \psi_i L_m(\psi_W), \quad i = 1, \ldots, m. \tag{5.2}$$

Note also that expressions (5.1) and (3.21) lead to

$$S_i(t, \lambda) = \lambda^r \psi_i L_{m,i}(\psi_W), \tag{5.3}$$

where $L_{m,i}$ are the ordinary differential operators of order $m - 1$, whose kernels are given by $\mathrm{Ker}(L_{m,i}) = \{\phi_1, \ldots, \widehat{\phi_i}, \ldots, \phi_m\}$. The action of $L_{m,i}$ is defined in Eq. (3.22)

Consider the KP hierarchy determined by the wave-eigenfunction $\psi_W(t, \lambda)$ with the associated hyperplane $\mathbf{W}$ in the universal Sato Grassmannian as in Eq. (2.18). Let $\{\phi_1, \ldots, \phi_m\}$ be m linearly independent functions. Let furthermore $S_i(t, \lambda)$ be m objects defined as in Definition 5.1 in terms of functions ψ_i dual to ϕ_i according to (3.8) and in terms of the wave-eigenfunction $\psi_W(t, \lambda)$ of the underlying KP hierarchy. The following main proposition establishes a connection between cKP $_{r,m}$ reduction of the KP hierarchy and the Grassmannian formulation.

Proposition 5.1. *The KP hierarchy associated to the hyperplane $\mathbf{W}$ in the universal Sato Grassmannian is of the reduced cKP$_{r,m}$ type if and only if the following two conditions:*

$$\partial_x S_i(t, \lambda) \in \mathbf{W}, \tag{5.4}$$

$$\sum_{i=0}^{m} c_i S_i(t, \lambda) \in \mathbf{W} \text{ implies } c_i = 0, \tag{5.5}$$

are satisfied for some m linearly independent functions $\{\phi_1, \ldots, \phi_m\}$. The correspond-
ing Lax operator is of the form $L = B_r + \sum_{i=1}^{m} \Phi_i D^{-1} \Psi_i$ with $\Phi_i = \phi_i$ and $\Psi_i(t) =$
$\int d\lambda \, \psi_W^*(t, \lambda) \, S_i(t_0, \lambda)$.

Proof. We start the proof with the cKP $_{r,m}$ system as given in (4.1) with both $\{\Phi_i\}$ and $\{\Psi_i\}$
being linearly independent set of functions and we are going to show that the conditions (5.4)
and (5.5) hold. We take $\Phi_i = \phi_i$. It follows from (4.3), (2.18) and $\partial_x S(\psi_W(t, \lambda), \Psi_i(t)) =$
$\psi_W(t, \lambda)\Psi_i(t)$ that

$$\lambda^r \psi_W^{(j)}(t, \lambda) - \sum_{i=1}^{m} \phi_i^{(j)}(t) \, S(\psi_W(t, \lambda), \Psi_i(t)) \in \mathbf{W}, \quad j = 0, \ldots, m-1, \qquad (5.6)$$

or in the matrix notation:

$$\lambda^r \begin{pmatrix} \psi_W(t, \lambda) \\ \psi_W^{(1)}(t, \lambda) \\ \vdots \\ \psi_W^{(m-1)}(t, \lambda) \end{pmatrix} - \mathcal{W}_{m \times m} \begin{pmatrix} S(\psi_W(t, \lambda), \Psi_1(t)) \\ S(\psi_W(t, \lambda), \Psi_2(t)) \\ \vdots \\ S(\psi_W(t, \lambda), \Psi_m(t)) \end{pmatrix} \in \mathbf{W} \qquad (5.7)$$

meaning that each element of the above combination of columns belongs to the Grassman-
nian $\mathbf{W}$. One finds easily from (5.7) and $\partial_x S(\psi_W(t, \lambda), \Psi_i(t)) = \psi_W(t, \lambda)\Psi_i(t)$ that

$$\lambda^r \partial_x \mathcal{W}_{m \times m}^{-1} \begin{pmatrix} \psi_W(t, \lambda) \\ \psi_W^{(1)}(t, \lambda) \\ \vdots \\ \psi_W^{(m-1)}(t, \lambda) \end{pmatrix} \in \mathbf{W}. \qquad (5.8)$$

In terms of the matrix elements $(\mathcal{W}_{m \times m}^{-1})_{ij}$ from (3.4), relation (5.8) takes a form

$$\lambda^r \partial_x \sum_{j=0}^{m-1} (\mathcal{W}_{m \times m}^{-1})_{ij} \psi_W^{(j)}(t, \lambda) \in \mathbf{W} \qquad (5.9)$$

for each $i = 1, \ldots, m$. This yields condition (5.4) due to Lemma 3.1.

Recalling Lemma 3.1, we see that (5.7) implies

$$S(\psi_W(t, \lambda), \Psi_i) - S_i(t, \lambda) \in \mathbf{W}, \quad i = 1, \ldots, m. \qquad (5.10)$$

If now it holds that

$$\sum_{i=1}^{m} c_i S_i(t, \lambda) \in \mathbf{W}, \qquad (5.11)$$

then because of relation (5.10), Eq. (5.11) implies that

$$\sum_{i=1}^{m} c_i S(\psi_W(t, \lambda), \Psi_i) \in \mathbf{W}, \qquad (5.12)$$

and therefore due to Eq. (2.17), $\sum_{i=1}^{m} c_i \Psi_i \lambda^{-1} \exp \xi(t, \lambda) = 0$. We note that from the linear independence of $\{\Psi_1, \ldots, \Psi_m\}$, it follows that $c_i = 0$, which is the desired result. Moreover, $\Psi_i(t) = \int d\lambda \, \psi_W^*(t, \lambda) \, \mathcal{S}_i(t_0, \lambda)$ holds due to (5.10).

From now on, we will assume that given is the system of the linearly independent functions $\{\phi_1, \ldots, \phi_m\}$ together with conditions (5.4) and (5.5). We are going to show that the KP hierarchy associated with the wave-eigenfunction $\psi_W(t, \lambda)$ satisfying constraints (5.4) and (5.5) belongs to the cKP $_{r,m}$ class.

We start by defining m functions:

$$\Psi_i(t, t_0) \equiv \int d\lambda \, \psi_W^*(t, \lambda) \, \mathcal{S}_i(t_0, \lambda), \quad i = 1, \ldots, m, \tag{5.13}$$

with $\mathcal{S}_i(t_0, \lambda)$ defined as in (5.1). First, it follows clearly from definition (5.13) that Ψ_i is an adjoint eigenfunction in the multi-time t. Secondly, Ψ_i is non-zero only for $\mathcal{S}_i(t_0, \lambda)$ not in $\mathbf{W}$ due to the Hirota's identity. According to the condition (5.5), the m functions Ψ_i are linearly independent.

What remains to be proven in order to establish that Ψ_i from Eq. (5.13) are adjoint eigenfunctions is that the functions Ψ_i do not depend on the second multi-parameter t_0. Indeed, from the condition (5.4), it follows immediately that $\partial_{x_0} \Psi_i(t, t_0) = 0$, and accordingly Ψ_i does not depend on $x_0 = (t_0)_1$. To complete the proof it remains to show that indeed $\partial \Psi_i / \partial (t_0)_n = 0$ for $n > 1$.

Define an oscillatory function $\psi_V(t, \lambda)$ by

$$L_m(\psi_W(t, \lambda)) \equiv \lambda^m \psi_V(t, \lambda). \tag{5.14}$$

Since $\psi_W(t, \lambda) = W \exp \xi(t, \lambda)$ where W is a dressing operator, we find that $\psi_V(t, \lambda) = V \exp \xi(t, \lambda)$ where $V = L_m W D^{-m}$ has like W a form of the dressing operator $V = 1 + \sum_{i=1}^{\infty} v_i D^{-i}$. Alternatively, we can rewrite $L_m = V D^m W^{-1}$. Consider $L_{m+r} \equiv V D^{m+r} W^{-1}$ such that $L_{m+r}(\psi_W(t, \lambda)) = \lambda^r L_m(\psi_W(t, \lambda))$. Since $\lambda^r L_m(\psi_W(t, \lambda)) \in \mathbf{W}$, the operator L_{m+r} is an ordinary differential operator. Moreover, we find that the KP Lax operator $L = W D^r W^{-1}$ can be written in terms of two ordinary differential operators as $L = L_m^{-1} L_{m+r}$. From [15–17], we know that the KP hierarchy equations $\partial_n L = [(L)_+^{n/r}, L]$ for $L = L_m^{-1} L_{m+r}$ are equivalent to the following flows on the differential operators L_m and L_{m+r}:

$$\partial_n L_m = (L_{m+r} L_m^{-1})_+^{n/r} L_m - L_m (L_m^{-1} L_{m+r})_+^{n/r}, \tag{5.15}$$

$$\partial_n L_{m+r} = (L_{m+r} L_m^{-1})_+^{n/r} L_{m+r} - L_{m+r} (L_m^{-1} L_{m+r})_+^{n/r}. \tag{5.16}$$

It has been shown in [10] that Eqs. (5.15) and (5.16) imply that $\phi_i \in \mathrm{Ker}\, L_m$ and $\psi_i \in \mathrm{Ker}\, L_m^*$ are "up to a gauge rotation" (adjoint) eigenfunctions satisfying

$$\partial_n \phi_i = (L_m^{-1} L_{m+r})_+^{n/r} (\phi_i), \quad i = 1, \ldots, m, \tag{5.17}$$

$$\partial_n \psi_i = -((L_{m+r} L_m^{-1})^*)_+^{n/r} (\psi_i), \quad i = 1, \ldots, m \tag{5.18}$$

We recognize in the above equations the setting of Lemma 3.2 with Eqs. (5.17) and (5.18) being special cases of (3.26) and (3.27). Especially, we may use the results of Lemma 3.3 and Eq. (3.41) to find

$$\partial_n S_i = \lambda^r \langle L_m(\psi_W) | \psi_i \rangle_{(L_{m+r} L_m^{-1})_+^{n/r}} = \lambda^r A_{n-1}(L_m(\psi_W)), \tag{5.19}$$

with A_{n-1} being $(n-1)$th order differential operator. Since $\lambda^r A_{n-1}(L_m(\psi_W)) \in \mathbf{W}$, it follows immediately that $\partial \Psi_i / \partial (t_0)_n = 0$ and indeed Ψ is a function of the multi-time t only.

Note that on basis of relation (3.19) the alternative form of definition (5.13) appears to be

$$\Psi_i(t) \equiv \sum_{j=0}^{m-1} (W_{m \times m}^{-1})_{ij}(t_0) \int d\lambda \, \lambda^r \psi_W^*(t, \lambda) \, \psi_W^{(j)}(t_0, \lambda). \tag{5.20}$$

Accordingly, using (3.4), we find (see also [18]):

$$\sum_{i=1}^{m} \phi_i(t_0) \Psi_i(t) = \int d\lambda \, \lambda^r \psi_W^*(t, \lambda) \, \psi_W(t_0, \lambda), \tag{5.21}$$

from which it follows that

$$\sum_{i=1}^{m} (\partial_n \phi_i(t_0) - B_n(\phi_i(t_0))) \Psi_i(t) = 0, \quad n = 1, 2, \ldots \tag{5.22}$$

or equivalently

$$\sum_{i=1}^{m} (\partial_n \phi_i(t_0) - B_n(\phi_i(t_0))) S_i(t, \lambda) \in \mathbf{W}. \tag{5.23}$$

From the last identity (5.23) and condition (5.5), we conclude that ϕ_i are eigenfunctions for $i = 1, \ldots, m$.

Recall that $L \psi_W(t, \lambda) = \lambda^r \psi_W(t, \lambda) = L_+(\psi_W(t, \lambda)) + L_-(\psi_W(t, \lambda))$ with the pseudo-differential part $L_-(\psi_W(t, \lambda)) \sim O(\lambda^{-1}) \exp \xi(t, \lambda)$. Inserting it back into Eq. (5.21), we find

$$\sum_{i=1}^{m} \phi_i(t_0) \Psi_i(t) = \int d\lambda \, \psi_W^*(t, \lambda) \, L_-(\psi_W(t_0, \lambda)). \tag{5.24}$$

From [3], we conclude that (5.24) implies

$$L_-(\psi_W(t_0, \lambda)) = \sum_{i=1}^{m} \phi_i(t_0) \frac{1}{\lambda} \psi_W(t_0, \lambda) \Psi_i(t_0 - [\lambda^{-1}]) \tag{5.25}$$

up to terms in $\mathbf{W}$. Equivalently, we can rewrite the last relation in the desired form

$$L_-(\psi_W(t, \lambda)) = \sum_{i=1}^{m} \phi_i(t) S(\psi_W(t, \lambda), \Psi_i(t)), \tag{5.26}$$

from which Eq. (4.1) follows due to the fact that the pseudo-differential operators act freely on the wave-functions as seen from (2.11). $\square$

6. Truncated KP hierarchy as cKP hierarchy

Let us consider the truncated KP hierarchy defined by the dressing operator W containing only finite number of terms. Let K be a positive order differential operator of the order N, such that $N > m$ and such that $W = K D^{-N}$. Accordingly, the corresponding Lax operator is

$$L_{tr} = K D^r K^{-1} = W D^r W^{-1}. \qquad (6.1)$$

Let f_i and g_i with $i = 1, \ldots, N$ be elements of the kernels $\operatorname{Ker} K$ and $\operatorname{Ker} K^*$, respectively. As shown in [14] the Wilson–Sato equations (2.4) for the hierarchy defined by the Lax operator (6.1) take a simple form for the elements of $\operatorname{Ker} K$:

$$\partial_n f_i = \partial_x^n f, \quad i = 1, \ldots, N. \qquad (6.2)$$

Recall that

$$K^{-1} = \sum_{i=1}^{N} f_i D^{-1} g_i, \qquad (6.3)$$

and consequently

$$
\begin{aligned}
\operatorname{Res}(K D^r K^{-1}) &= \sum_{i=1}^{N} K D^r (f_i) g_i \\
&= \sum_{i=1}^{N} (-1)^{N-i} \frac{W[f_1, \ldots, f_N, f_i^{(r)}] W[f_1, \ldots, \widehat{f_i}, \ldots, f_N]}{W^2[f_1, \ldots, f_N]} \\
&= \partial_x \sum_{i=1}^{N} (-1)^{N-i} \frac{W[f_1, \ldots, \widehat{f_i}, \ldots, f_N, f_i^{(r)}]}{W[f_1, \ldots, f_N]} \\
&= \partial_x \partial_r \ln W[f_1, \ldots, f_N],
\end{aligned}
\qquad (6.4)
$$

where we used the Jacobi identity

$$
\begin{aligned}
&W[W[f_1, \ldots, f_m, g], W[f_1, \ldots, f_m, h]] \\
&= W[f_1, \ldots, f_m] W[f_1, \ldots, f_m, g, h],
\end{aligned}
\qquad (6.5)
$$

and (6.2).

Hence we reproduced the well-known result that the tau function for the truncated KP hierarchy is the Wronskian $\tau_{trun} = W[f_1, \ldots, f_N]$.

Due to (6.2), we can rewrite f_i as $f_i = \int dz\, \tilde{f}_i(z) \exp(\xi(t, z))$. Notice that

$$K \exp(\xi(t, \lambda)) = \lambda^N \psi_W(t, \lambda) \qquad (6.6)$$

due to the fact that $W = K D^{-N}$ is the dressing operator of the truncated hierarchy. It holds therefore that

$$0 = K(f_i) = \int dz\, z^N \tilde{f}_i(z) \psi_W(t, z). \qquad (6.7)$$

Accordingly, for any positive differential operator B, we find

$$\int dz \, z^N \, \tilde{f}_i(z) B(\psi_W(t, z)) = B\left(\int dz \, z^N \, \tilde{f}_i(z) \psi_W(t, z)\right) = 0 \qquad (6.8)$$

We now investigate the condition for L_{tr} to be within $cKP_{r,m}$ in the non-trivial case $N > m$. As shown above, the necessary condition for this to happen is that $\lambda^r L_m(\psi_W) \in \mathbf{W}$ or that there exists a positive differential operator B such that $\lambda^r L_m(\psi_W) = B(\psi_W)$. Comparing with (6.8), we find that $\lambda^r L_m(\psi_W) \in \mathbf{W}$ translates into

$$0 = \int dz \, z^N \, \tilde{f}_i(z) z^r L_m(\psi_W(t, z)) = L_m\left(\int dz \, \tilde{f}_i(z) z^{r+N} \psi_W(t, z)\right)$$

$$= L_m\left(\int dz \, \tilde{f}_i(z) z^r K e^{\xi(t, z)}\right) = L_m K D^r(f_i) \qquad (6.9)$$

for all $i = 1, \ldots, N$.

Hence $K D^r(f_i) \in \text{Ker}(L_m)$ for $i = 1, \ldots, N$. In [9] this condition was rewritten using a Jacobi identity for Wronskians as

$$W_{N+m+1}[f_1, \ldots, f_N, f_{i_1}^{(r)}, \ldots, f_{i_{m+1}}^{(r)}] = 0. \qquad (6.10)$$

7. Concluding remarks

We have seen that formulating the constrained KP hierarchy within the Sato Grassmannian becomes transparent when use is being made of the underlying ordinary differential operators with convenient parametrization of their kernels. Useful insight has been obtained by relating the notions of kernel elements of the underlying ordinary differential operators with that of the eigenfunctions of the KP hierarchy. The related connection of the bilinear concomitant introducing the canonical pairing structure on the space kernels to that of the squared eigenfunction potentials has then arisen naturally.

Let us complete our discussion by the following additional comments addressing fundamental questions of the formalism.

Remark 1. Due to relation (3.19) we have $\sum_{i=1}^{m} \Phi_i S_i = \lambda^r \psi_W$. Hence condition (5.5) can be understood as an obstruction to the usual KdV reduction with $\lambda^r \psi_W \in \mathbf{W}$.

Remark 2. Alternatively to (5.5) we could have expressed the relevant assumption in terms of the integrals:

$$\sum_{i=0}^{m} c_i \int d\lambda \, \psi_W^*(t, \lambda) \, S_i(t_0, \lambda) = 0 \text{ implies } c_i = 0. \qquad (7.1)$$

instead of involving the Sato Grassmannian $\mathbf{W}$ in (5.5). The arguments used in the proof would then have worked with small adjustments but without any need of making an additional assumption that $\int d\lambda \, \psi_W^*(t, \lambda) F(\psi_W(t', \lambda)) = 0$ implies $F(\psi_W(t', \lambda)) \in \mathbf{W}$ for $F(\psi_W(t', \lambda))$ of the positive order.

Remark 3. From definitions (5.14) and (5.15) we find that the flows of $\psi_V(t, \lambda)$ are given by

$$\partial_n \psi_V(t, \lambda) = (L_{m+r} L_m^{-1})_+^{n/r} (\psi_V(t, \lambda)). \qquad (7.2)$$

Based on (3.27) and (7.2) it makes now sense to define the squared eigenfunction potential $S(\psi_V, \psi_i)$ for $\psi_V(t, \lambda)$ and ψ_i with the following useful property:

$$S_i(t, \lambda) = \lambda^{r+m} S(\psi_V, \psi_i) \qquad (7.3)$$

Due to Eq. (7.3) one can rewrite Eq. (5.13) as

$$\Psi_i(t) = \int d\lambda \, \lambda^{r+m} \psi_W^*(t, \lambda) \, S(\psi_V(t_0, \lambda), \psi_i(t_0)), \quad i = 1, \ldots, m. \qquad (7.4)$$

Since $\partial_{x_0} \Psi_i(t, t_0) = 0$ it holds that $\int d\lambda \, \lambda^{r+m} \, \psi_W^*(t, \lambda) \, \psi_V(t_0, \lambda) = 0$.

Furthermore it is easy to see that $\lambda^{r+m} \psi_W^*(t, \lambda) = L_{m+r}^*(\psi_V^*(t, \lambda))$ as follows from definition (2.7) and $\psi_V^*(t, \lambda) = V^{*-1} \exp -\xi(t, \lambda)$ together with $L_{m+r}^* = W^{*-1}(-D)^{m+r} V^*$. Plugging it back in (7.4) we obtain the expression for Ψ_i in terms of ψ_i:

$$\Psi_i(t) = \int d\lambda \, L_{m+r}^*(\psi_V^*(t, \lambda)) \, S(\psi_V(t_0, \lambda), \psi_i(t_0)) = L_{m+r}^*(\psi_i). \qquad (7.5)$$

Remark 4. According to (5.2), the inclusion $\partial_x S_i \in \mathbf{W}$ can be rewritten as $\lambda^r L_m(\psi_W) \in \mathbf{W}$, where L_m is an m-order differential operator whose action on the wave function ψ_W can be viewed as m successive Darboux–Bäcklund transformations. With the kernel of L_m being $\{\Phi_1, \ldots, \Phi_m\}$, let w_j be such that $\Phi_j = \int d\lambda \, \lambda^{-1}(\psi_W w_j)$ for $j = 1, \ldots, m$. Accordingly, w_j are orthogonal to the subspace $\mathbf{W}' \equiv \text{span}\{L_m(\psi_W)\}$ of $\mathbf{W}$ with respect to the inner product $\langle u | v \rangle \equiv \int d\lambda \, \lambda^{-1} uv$. Hence, the inclusion $\lambda^r \mathbf{W}' \in \mathbf{W}$ is a co-dimension m inclusion. Also, condition (5.5) in view of (5.3) expresses the assumption about the codimension m being optimal. This establishes link to the formalism of [6,7].

Acknowledgements

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